

A relation between the shape of a permutation and the shape of the base poset derived from the Lehmer codes

Masaya Tomie

University of Morioka, Takizawa-mura, Iwate 020-0183, Japan (e-mail: tomie@morioka-u.ac.jp)

Abstract

For a permutation $\omega \in S_n$ Denoncourt constructed a poset M_ω which is the set of join-irreducibles of Lehmer codes of the permutations in $[e, \omega]$ in the inversion order on S_n . In this paper we show that M_ω is a B_2 -free poset if and only if ω is a 3412 – 3421-avoiding permutation.

1 Introduction

Let P, Q be a poset. A subposet $R \subset P$ is called a Q -pattern subposet if $R \simeq Q$ as a poset. We say that P is Q -free if P has no Q -pattern subposets. The number of 1 + 3-free and 2 + 2-free posets with n elements is $\frac{1}{n+1} \binom{2n}{n}$ the n -th Catalan number [5].

For $\sigma \in S_n, \pi \in S_k$ with $k < n$, we say a permutation σ is a π avoiding permutation if $st(\sigma(i_1)\sigma(i_2) \cdots \sigma(i_k)) \neq \pi(1)\pi(2) \cdots \pi(k)$ for any $1 \leq i_1 < i_2, \cdots < i_k \leq n$. The number of π avoiding permutation in S_n is $\frac{1}{n+1} \binom{2n}{n}$ for all $\pi \in S_3$ [3] [4]. In this paper we consider the relation B_2 -free posets where B_2 is Boolean algebra of rank 2 and 3412 – 3421-avoiding permutations by considering Lehmer codes.

In [2] Denoncourt showed that the set of Lehmer codes for permutations in Λ_ω ordered by the product order on \mathbb{N}^n is a distributive lattice where $\Lambda_\omega = \{\sigma | \text{Inv}(\sigma) \subset \text{Inv}(\omega)\}$ and he also gave the expression of M_ω which is the set of join-irreducibles of the set of Lehmer codes for Λ_ω .

In this paper we focus on the relation between the shape of M_ω and that of ω and obtain the following result.

Theorem 1.1. *M_ω is a B_2 -free poset if and only if ω is a 3412 – 3421-avoiding permutation.*

2 Notations and Remarks

In this paper we use 1-line notation, this is $\omega = \omega(1)\omega(2) \cdots \omega(n)$ for $\omega \in S_n$. Put $\Lambda_\omega := \{\sigma | \text{Inv}(\sigma) \subset \text{Inv}(\omega)\}$ where $\text{Inv}(\omega) := \{(i, j) | 1 \leq i < j \leq n, \omega(i) > \omega(j)\}$. In other words Λ_ω is the interval $[e, \omega]$ in the left Bruhat order.

We put $c_i(\omega) := \#\{j | 1 \leq i < j \leq n, \omega(i) > \omega(j)\}$ the number of inversions of ω with first coordinate is i and $c_{ij}(\omega) := \#\{k | 1 \leq i < k < j \leq n, \omega(i) > \omega(k)\}$ the number of inversions of ω with first coordinate is i and second coordinate is between i and j . The finite sequence

$$\mathbf{c}(\omega) := (c_1(\omega), c_2(\omega), \dots, c_n(\omega))$$

is called the *Lehmer code* for ω and let $\mathbf{c}(\Lambda_\omega)$ be the set of Lehmer codes of permutations in Λ_ω . In [2] Denoncourt showed the following result.

Theorem 2.1 (Denoncourt). *For $\omega \in S_n$ the subposet $\mathbf{c}(\Lambda_\omega)$ of \mathbb{N} is a distributive lattice.*

Let L be a finite distributive lattice and P the subposet of join irreducible elements in L . Then the *fundamental theorem for finite distributive lattices* states that $L \simeq J(P)$ where $J(P)$ is the poset of order ideals of P ordered by inclusion [1]. In [2] Denoncourt determined the set of join irreducible elements in $\mathbf{c}(\Lambda_\omega)$.

Definition 2.1 (Denoncourt). *For $i \in [n]$ such that $c_i(\omega) > 0$ and for each $x \in [c_i(\omega)]$, define $m_{i,x}(\omega) \in \mathbb{N}$ coordinate-wise by*

1. $\pi_j(m_{i,x}(\omega)) = 0$ if $(i, j) \in \text{Inv}(\omega)$,
2. $\pi_j(m_{i,x}(\omega)) = 0$ if $j < i$,
3. $\pi_j(m_{i,x}(\omega)) = x$ if $j = i$,
4. $\pi_j(m_{i,x}(\omega)) = \max\{0, x - c_{i,j}(\omega)\}$ if $j > i$ and $(i, j) \notin \text{Inv}(\omega)$

where $\pi_j(m_{i,x}(\omega))$ denotes the j -th coordinate of $m_{i,x}(\omega)$. Put $M_\omega = \{m_{i,x}(\omega) | 1 \leq i \leq n, c_i(\omega) > 0, x \in [c_i(\omega)]\}$.

Let $M_\omega = \{m_{i,x}(\omega) | 1 \leq i \leq n \text{ such that } c_i(\omega) > 0 \text{ and } x \in [c_i(\omega)]\}$ and $C_i(\omega) = \{m_{i,x}(\omega) | x \in [c_i(\omega)]\}$ for $1 \leq i \leq n$ such that $c_i(\omega) \neq 0$. Then M_ω is a subposet of \mathbb{N}^n in the product order. Denoncourt showed the following results, see Corollary 5.6 and Theorem 6.6 of his paper [2].

Theorem 2.2 (Denoncourt). *The set M_ω is the set of join irreducible elements of $\mathbf{c}(\Lambda_\omega)$.*

Lemma 2.1 (Denoncourt). *1. For $\omega \in S_n$ and $1 \leq i < j \leq n$ with $(i, j) \in \text{Inv}(\omega)$, every element of $C_i(\omega)$ is incomparable with every element of $C_j(\omega)$,*

2. *For $\omega \in S_n$ and $1 \leq i < j \leq n$ with $(i, j) \notin \text{Inv}(\omega)$, we have $m_{i,x}(\omega) > m_{j,y}(\omega)$ if and only if $y \leq x - c_{i,j}(\omega)$.*

In other words there exists a pair of comparable elements $m_{i,x}(\omega) > m_{j,y}(\omega)$ with $1 \leq i < j \leq n$ if and only if $st(\omega(i)\omega(j)\omega(l)) = 231$ for some $j < l$. Hence we have the following corollary.

Corollary 2.1. *If ω is a 231-avoiding permutation then M_ω is disjoint union of the chains.*

3 Main Result

In this section we give a proof of the following result.

Theorem 3.1. M_ω is a B_2 -free poset if and only if ω is a 3412 – 3421-avoiding permutation.

Definition 3.1. Let P be a poset. A subposet $\{a, b, c, d\} \subset P$ with distinct elements is called B_2 -pattern subposet if $\{a, b, c, d\} \simeq B_2$ where B_2 is a Boolean algebra of rank 2. We say that P has a B_2 -pattern if P has a B_2 -pattern subposet.

We will define the following poset patterns.

Definition 3.2. 1. For $1 \leq i < j \leq n, b < a \in [c_i(\omega)]$ and $c < d \in [c_j(\omega)]$ with $a + c = b + d$ the poset $\{m_{i,a}(\omega), m_{i,b}(\omega), m_{j,c}(\omega), m_{j,d}(\omega)\}$ is called *parallelogram – pattern poset* if $m_{i,a}(\omega) > m_{j,d}(\omega), m_{i,b}(\omega) > m_{j,c}(\omega)$ and the two elements $m_{i,b}(\omega)$ and $m_{j,d}(\omega)$ are incomparable.

We say that M_ω has a *parallelogram – pattern* if M_ω contains a *parallelogram – pattern poset*.

2. For $1 \leq i < j \leq n, b < a \in [c_i(\omega)]$ and $c < d \in [c_j(\omega)]$ with $a + c = b + d$ the poset $\{m_{i,a}(\omega), m_{i,b}(\omega), m_{j,c}(\omega), m_{j,d}(\omega)\}$ is called C_4 – *parallelogram – pattern poset* if $m_{i,a}(\omega) > m_{j,d}(\omega), m_{i,b}(\omega) > m_{j,c}(\omega)$ and the two elements $m_{i,b}(\omega)$ and $m_{j,d}(\omega)$ are comparable. If $m_{i,b}(\omega)$ and $m_{j,d}(\omega)$ are comparable then we have $m_{i,b}(\omega) > m_{j,d}(\omega)$ because the i -th entry of $m_{i,b}(\omega)$ is b and that of $m_{j,d}(\omega)$ equals to 0.

We say that M_ω has a C_4 – *parallelogram – pattern* if M_ω contains a C_4 – *parallelogram – pattern poset*. Especially we have $m_{i,a}(\omega) > m_{i,b}(\omega) > m_{j,d}(\omega) > m_{j,c}(\omega)$.

Figure 1 shows the shape of the parallelogram-pattern and the C_4 -parallelogram pattern.

A parallelogram-pattern subposet is also B_2 -pattern subposet, but a B_2 -pattern subposet is not always a parallelogram-pattern subposet.

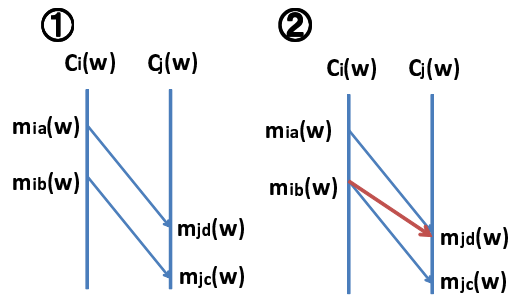


Figure 1: Part.1 is the parallelogram-pattern and Part.2 is the C_4 -parallelogram-pattern.

The following statement is useful but it is easy to see, hence we omit the proof.

Lemma 3.1. For $1 \leq i < j \leq n$ and $\omega \in S_n$,

1. if $m_{i,a}(\omega) > m_{j,b}(\omega)$ with $a, b \geq 2$ then $m_{i,(a-1)}(\omega) > m_{j,(b-1)}(\omega)$,
2. if $m_{i,a}(\omega) > m_{j,b}(\omega)$ with $a < c_i(\omega)$ and $b < c_j(\omega)$ then $m_{i,(a+1)}(\omega) > m_{j,(b+1)}(\omega)$.

Figure 2 shows a visualization of Lemma 3.1.

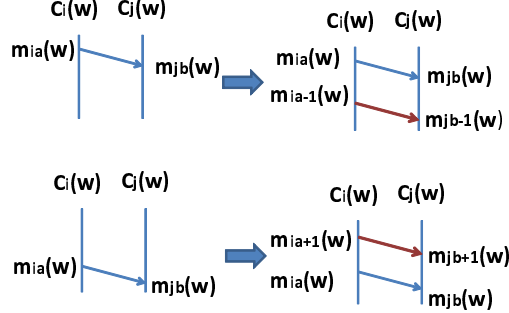


Figure 2:

Lemma 3.2. For $1 \leq i < j \leq n$ and $\omega \in S_n$ if $m_{i,p}(\omega) > m_{j,q}(\omega)$ for some $1 \leq p \leq c_i(\omega)$ and $1 \leq q \leq c_j(\omega)$ then $\{k | j < k, \omega(i) > \omega(k)\} \subset \{l | j < l, \omega(j) > \omega(l)\}$.

Proof. By Lemma 2.1 we have $\omega(i) < \omega(j)$ because $m_{i,p}(\omega) > m_{j,q}(\omega)$ for some $1 \leq p \leq c_i(\omega)$ and $1 \leq q \leq c_j(\omega)$. Hence we have $\{k | j < k, \omega(i) > \omega(k)\} \subset \{l | j < l, \omega(j) > \omega(l)\}$. □

Lemma 3.3. If $m_{i,p}(\omega) > m_{j,q}(\omega)$ for some $\omega \in S_n$, $p \in [c_i(\omega)]$ and $q \in [c_j(\omega)]$, then we have $c_j(\omega) \geq c_i(\omega) + q - p$.

Proof. Set $m_{i,p}(\omega) = (0, \dots, 0, \overbrace{p}^i, \dots, \overbrace{x}^j, \dots)$, $m_{j,q}(\omega) = (0, \dots, 0, \overbrace{0}^i, \dots, \overbrace{q}^j, \dots)$. Then we have $x = p - c_{i,j}(\omega)$ and $c_{i,j}(\omega) \leq p - q$ because $m_{i,p}(\omega) > m_{j,q}(\omega)$ and $x \leq q$. Also we have $c_i(\omega) = c_{i,j}(\omega) + \#\{k | j < k, \omega(i) > \omega(k)\} \leq c_{i,j}(\omega) + c_j(\omega) \leq p - q + c_j(\omega)$. Hence we have $c_j(\omega) \geq c_i(\omega) + q - p$. □

The above Lemma 3.3 says that if $m_{i,p}(\omega) > m_{j,q}(\omega)$ with $1 \leq i < j \leq n$, $p \in [c_i(\omega)]$ and $q \in [c_j(\omega)]$ then there exists $c_i(\omega) + q - p \in [c_j(\omega)]$ such that $m_{i,c_i(\omega)}(\omega) > m_{j,c_i(\omega)+q-p}(\omega)$ by Lemma 3.1. Figure 3 shows a visualization of Lemma 3.3.

For root poset M_ω we have the following observation.

Lemma 3.4. For $\omega \in S_n$ if M_ω has a C_4 -parallelogram-pattern then M_ω has a parallelogram-pattern.

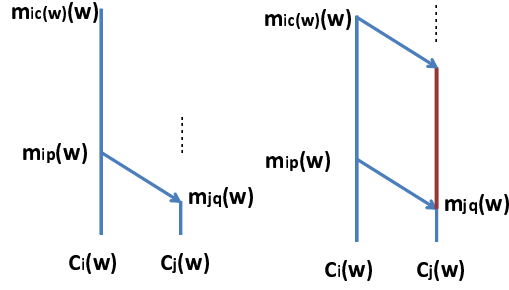


Figure 3:

Proof. By assumption there exists $\{m_{i,a}(\omega), m_{i,b}(\omega), m_{j,c}(\omega), m_{j,d}(\omega)\} \subset M_\omega$ such that $m_{i,a}(\omega) > m_{i,b}(\omega) > m_{j,d}(\omega) > m_{j,c}(\omega)$ for some $1 \leq i < j \leq n, b < a \in [c_i(\omega)]$ $c < d \in [c_j(\omega)]$ with $a + c = b + d$. It is easy to see that $b \geq d$ because $m_{i,b}(\omega) > m_{j,d}(\omega)$.

We will show that it is possible to construct a parallelogram-pattern poset from $\{m_{i,a}(\omega), m_{i,b}(\omega), m_{j,c}(\omega), m_{j,d}(\omega)\}$ by induction on $b - d$.

If $b = d$ then $c_i(\omega) \geq a \geq b + 1 > d$ and by Lemma 3.3 we have $d + 1 \leq c_j(\omega)$.

Consider the subposet $\{m_{i,a}(\omega), m_{i,b}(\omega), m_{j,c+1}(\omega), m_{j,d+1}(\omega)\} \subset M_\omega$ where i, j, a, b, c, d are as above.

We have $m_{i,a}(\omega) > m_{j,d+1}(\omega)$ because $m_{i,b}(\omega) > m_{j,d}(\omega)$ and hence $m_{i,a}(\omega) \geq m_{i,b+1}(\omega) > m_{j,d+1}(\omega)$. Also $m_{i,b}(\omega) > m_{j,c+1}(\omega)$ because $m_{i,b}(\omega) > m_{j,d}(\omega) \geq m_{j,c+1}(\omega)$. Obviously $m_{i,b}(\omega)$ and $m_{j,d+1}(\omega)$ are incomparable. Therefore the subposet $\{m_{i,a}(\omega), m_{i,b}(\omega), m_{j,c+1}(\omega), m_{j,d+1}(\omega)\}$ is parallelogram-pattern poset.

Assume that we can construct a parallelogram-pattern poset from $\{m_{i,a}(\omega), m_{i,b}(\omega), m_{j,c}(\omega), m_{j,d}(\omega)\}$ for $b - d \leq k - 1$.

If $b - d = k$ then we consider a subposet $\{m_{i,a}(\omega), m_{i,b}(\omega), m_{j,c+1}(\omega), m_{j,d+1}(\omega)\} \subset M_\omega$ where i, j, a, b, c, d are as above. We have $m_{i,a}(\omega) > m_{j,d+1}(\omega)$ because $m_{i,b}(\omega) > m_{j,d}(\omega)$ and hence $m_{i,a}(\omega) \geq m_{i,b+1}(\omega) > m_{j,d+1}(\omega)$. Also $m_{i,b}(\omega) > m_{j,c+1}(\omega)$ because $m_{i,b}(\omega) > m_{j,d}(\omega) \geq m_{j,c+1}(\omega)$. If $m_{i,b}(\omega) > m_{j,d+1}(\omega)$ then we can construct a parallelogram-pattern poset by the assumption because $b - (d + 1) = k - 1$. If $m_{i,b}(\omega)$ and $m_{j,d+1}(\omega)$ are incomparable then the poset $\{m_{i,a}(\omega), m_{i,b}(\omega), m_{j,c+1}(\omega), m_{j,d+1}(\omega)\}$ is a parallelogram-pattern poset. This completes the proof. \square

Lemma 3.5. For $\omega \in S_n$ the poset M_ω has a B_2 -pattern if and only if it has a parallelogram-pattern.

Proof. If M_ω has a parallelogram-pattern then obviously it has a B_2 -pattern. Conversely we assume that M_ω has a B_2 -pattern.

Let $\{m_{i,a}(\omega), m_{j,b}(\omega), m_{k,c}(\omega), m_{l,d}(\omega)\}$ with $i, j, k, l \in \mathbb{N}, a \in [c_i(\omega)], b \in [c_j(\omega)], c \in [c_k(\omega)]$ and $d \in [c_l(\omega)]$ be a B_2 -pattern subposet of M_ω where $m_{i,a}(\omega)$ (resp. $m_{l,d}(\omega)$) is the maximum (resp. minimum) element and $m_{j,b}(\omega)$ and $m_{k,c}(\omega)$ are incomparable. We can set $j < k$ without loss of generality and hence we have $i \leq j < k \leq l$.

Case.1 (The case of $d \geq 2$)

We have $m_{i,a-d-1}(\omega) > m_{l,1}(\omega)$ because $m_{i,a}(\omega) > m_{l,d}(\omega)$ and by Lemma 3.1. Hence the poset $\{m_{i,a}(\omega), m_{i,a-d+1}(\omega), m_{l,d}(\omega), m_{l,1}(\omega)\}$ is either a parallelogram-pattern poset or a C_4 -parallelogram-pattern poset. By Lemma 3.4 the poset M_ω has a parallelogram-pattern for both cases.

Case.2 (The case of $d = 1$ and $i = j$)

We have $c_l(\omega) \geq c_i(\omega) + 1 - b \geq a - b + 1$ because $m_{i,b}(\omega) > m_{l,1}(\omega)$ and by Lemma 3.3. Also we have $a - b + 1 \geq 2$ and $c_l(\omega) \geq 2$ because $i = j$ and $m_{i,a}(\omega) > m_{j,b}(\omega)$.

Hence the poset $\{m_{i,b+1}(\omega), m_{i,b}(\omega), m_{l,2}(\omega), m_{l,1}(\omega)\}$ is either a parallelogram-pattern poset or a C_4 -parallelogram-pattern poset because $m_{i,b}(\omega) > m_{l,1}(\omega)$ and by Lemma 3.1.

By Lemma 3.4 the poset M_ω has a parallelogram-pattern for both cases.

Case.3 (The case of $d = 1$ and $k = l$)

In this case we have $m_{i,a}(\omega) > m_{k,c}(\omega) \geq m_{k,d}(\omega)$ so we have $c > d \geq 1$.

By Lemma 3.1 the poset $\{m_{i,a}(\omega), m_{i,a-1}(\omega), m_{k,c}(\omega), m_{k,c-1}(\omega)\}$ is either a parallelogram-pattern poset or a C_4 -parallelogram-pattern poset. By Lemma 3.4 the poset M_ω has a parallelogram-pattern for both cases.

Next we will consider the case of $d = 1$ with $i < j < k < l$.

Case.4 (The case of $b \geq 2$ or $c \geq 2$ with $d = 1$ and $i < j < k < l$)

We will consider the case of $b \geq 2$ and for the case of $c \geq 2$ we can use the same argument. Because $m_{i,a}(\omega) > m_{j,b}(\omega)$ we have $a \geq 2$ and $m_{i,a-1}(\omega) > m_{j,b-1}(\omega)$ by Lemma 3.1

Hence the poset $\{m_{i,a}(\omega), m_{i,a-1}(\omega), m_{j,b}(\omega), m_{j,b-1}(\omega)\}$ is either a parallelogram-pattern poset or a C_4 -parallelogram-pattern poset. By Lemma 3.4 the poset M_ω has a parallelogram-pattern for both cases.

Case.5 (The case of $b = c = d = 1$ with $i < j < k < l$ and $c_j(\omega) \geq 2$ or $c_k(\omega) \geq 2$)

We will consider the case of $c_j(\omega) \geq 2$ and for the case of $c_k(\omega)$ we can use the same argument.

From Lemma 3.3 we obtain $c_l(\omega) \geq c_j(\omega) + 1 - 1 \geq 2$ because $m_{j,1}(\omega) > m_{l,1}(\omega)$ and $c_j(\omega) \geq 2$. By Lemma 3.1 we have $m_{j,2}(\omega) > \exists m_{l,2}(\omega)$. Hence the poset $\{m_{j,2}(\omega), m_{j,1}(\omega), m_{l,2}(\omega), m_{l,1}(\omega)\}$ is either a parallelogram-pattern poset or a C_4 -parallelogram-pattern poset. By Lemma 3.4 the poset M_ω has a parallelogram-pattern for both cases.

Case.6 (The case of $b = c = d = 1$ with $i < j < k < l$ and $c_j(\omega) = c_k(\omega) = 1$)

We have $\omega(i) < \omega(j)$ and $\omega(k) < \omega(l)$ because $m_{i,a}(\omega) > m_{j,b}(\omega)$ and $m_{k,c}(\omega) > m_{l,d}(\omega)$.

Set $m_{i,a}(\omega) = (0, \dots, 0, \overbrace{a}^i, \dots, \overbrace{x}^l, \dots)$ and $m_{l,1}(\omega) = (0, \dots, 0, \overbrace{0}^i, \dots, 0, \overbrace{1}^l, \dots)$. We obtain $x \geq 1$ so there exists $p > l$ such that $\omega(i) > \omega(p)$. It is easy to see that $\{y | j < y, \omega(j) > \omega(y)\} = \{p\}$ because $c_j(\omega) = 1$. Hence $\omega(j) < \omega(k)$ and then we get $\omega(i) < \omega(j) < \omega(k) < \omega(l)$.

Now we have $m_{j,1}(\omega) = (0, \dots, 0, \overbrace{1}^j, 1, \dots, 1, \overbrace{1}^k, 1, \dots, 1, \overbrace{1}^l, 1, \dots, 1, \overbrace{0}^p, 0, \dots, 0)$ and $m_{k,1}(\omega) = (0, \dots, 0, \overbrace{0}^j, 0, \dots, 0, \overbrace{1}^k, 1, \dots, 1, \overbrace{1}^l, 1, \dots, 1, \overbrace{0}^p, 0, \dots, 0)$ because $c_k(\omega) = 1$ and

$\omega(k) > \omega(i) > \omega(p)$. Then we get $m_{j,1}(\omega) > m_{k,1}(\omega)$ and this contradicts the assumption that $m_{j,1}(\omega)$ and $m_{k,1}(\omega)$ are incomparable. Therefore the **case.6** never happens.

This completes the proof. \square

Lemma 3.6. *For $\omega \in S_n$ the poset M_ω has a parallelogram-pattern if and only if ω has a 3412-pattern or a 3421-pattern.*

Proof. Suppose that ω has a 3412-pattern. Then there exists $i < j < k < l$ such that $st(\omega(i)\omega(j)\omega(k)\omega(l)) = 3412$ and we obtain $c_i(\omega) \geq 2$ and $c_j(\omega) \geq 2$. We have

$$\begin{aligned} m_{i,c_i(\omega)}(\omega) &= (0, \dots, 0, \overbrace{c_i(\omega)}^i, \dots, \overbrace{p}^j, \dots) \\ m_{i,c_i(\omega)-1}(\omega) &= (0, \dots, 0, \overbrace{c_i(\omega)-1}^i, \dots, \overbrace{p-1}^j, \dots) \\ m_{j,2}(\omega) &= (0, \dots, 0, \overbrace{0}^i, \dots, 0, \overbrace{2}^j, \dots) \\ m_{j,1}(\omega) &= (0, \dots, 0, \overbrace{0}^i, \dots, 0, \overbrace{1}^j, \dots) \end{aligned}$$

with $p \geq 2$ because $\omega(i) > \omega(k), \omega(l)$.

Claim

$$m_{i,c_i(\omega)-1}(\omega) > m_{j,1}(\omega)$$

For $x \leq j$ the x -th entry of $m_{j,1}(\omega)$ is less than that of $m_{i,c_i(\omega)-1}(\omega)$.

For $x > j$ the x -th entry of $m_{j,1}(\omega)$ is 0 or 1. If that of $m_{j,1}(\omega)$ equals to 1 then $(j, y) \notin \text{Inv}(\omega)$ for $j < y \leq x$ and hence $\omega(i) < \omega(j) < \omega(x)$. Therefore $(i, y) \notin \text{Inv}(\omega)$ for $j < y \leq x$ and the x -th entry of $m_{i,c_i(\omega)-1}(\omega)$ is $(p-1) \geq 1$. The x -th entry of $m_{j,1}(\omega)$ is less than that of $m_{i,c_i(\omega)-1}(\omega)$ if that of $m_{j,1}(\omega)$ is 0. Hence we have $m_{i,c_i(\omega)-1}(\omega) > m_{j,1}(\omega)$.

For the set $\{m_{i,c_i(\omega)}(\omega), m_{i,c_i(\omega)-1}(\omega), m_{j,2}(\omega), m_{j,1}(\omega)\}$ we obtain $m_{i,c_i(\omega)}(\omega) > m_{i,c_i(\omega)-1}(\omega) > m_{j,1}(\omega)$. Also we have $m_{i,c_i(\omega)}(\omega) > m_{j,2}(\omega) > m_{j,1}(\omega)$ by Lemma 3.1.

Then the induced subposet is either a parallelogram-pattern poset or a C_4 -parallelogram-pattern poset. By Lemma 3.4 the poset M_ω has a parallelogram-pattern for both cases. We can use the same argument if ω has a 3421-pattern.

Suppose that M_ω has a parallelogram-pattern poset $\{m_{i,a}(\omega), m_{i,b}(\omega), m_{j,c}(\omega), m_{j,d}(\omega)\}$ with $1 \leq i < j \leq n, b < a \in [c_i(\omega)]$ and $d < c \in [c_j(\omega)]$ and $a + d = b + c$ where $m_{i,a}(\omega)$ (resp. $m_{j,d}(\omega)$) is the maximum (resp. minimum) element and $m_{i,b}(\omega)$ and $m_{j,c}(\omega)$ are incomparable. In particular $c \geq 2$ because $c > d \geq 1$.

From Lemma 2.1 we have $\omega(i) < \omega(j)$. Put

$$\begin{aligned} m_{i,a}(\omega) &= (0, \dots, 0, \overbrace{a}^i, \dots, \overbrace{x}^j, \dots) \\ m_{j,c}(\omega) &= (0, \dots, 0, \overbrace{0}^i, \dots, \overbrace{c}^j, \dots) \end{aligned}$$

where $x \geq c \geq 2$ hence $\#\{y | j < y, \omega(i) > \omega(y)\} \geq 2$ and there exists $j < y_1 < y_2$ such that $\omega(i) > \omega(y_1)$ and $\omega(i) > \omega(y_2)$. Then we have $st(\omega(i)\omega(j)\omega(y_1)\omega(y_2)) = 3412$ or 3421 .

This completes the proof. □

From Lemma 3.5 and Lemma 3.6 we obtain the following result.

Theorem 3.2. *M_ω is a B_2 -free poset if and only if ω is a 3412 – 3421-avoiding permutation.*

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